Section 2.2: The Algebra of Functions

Objectives

- Find the sum, difference, product or quotient of two functions and the corresponding domain
- Find the composition of functions
- Calculate the difference quotient

Arithmetic Operations on Functions

Sum: $(f + g)(x) = f(x) + g(x)$

Difference: $(f - g)(x) = f(x) - g(x)$

Product: $(fg)(x) = f(x) \cdot g(x)$

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Example: Let f and g be two functions defined as $f(x) = 2x - 1$ and $g(x) = x + 3$. Find the following and determine the domain of each:

$(f+g)(x) = f(x) + g(x) = 2x - 1 + x + 3 = 3x + 2$

	domain
$f(x)$	$(-\infty, \infty)$
$g(x)$	$(-\infty, \infty)$
$(f+g)(x)$	$(-\infty, \infty)$

Domain: $(-\infty, \infty)$

$(f-g)(x) = f(x) - g(x) = 2x - 1 - (x + 3)$

$= 2x - 1 - x - 3$

$= x - 4$

Domain: $(-\infty, \infty)$

$(fg)(x) = f(x)g(x) = (2x-1)(x+3) = 2x^2 + 6x - x - 3 = 2x^2 + 5x - 3$

FOIL

Domain: $(-\infty, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x-1}{x+3}$

Denominator $\neq 0$

Can't plug in -3.

-3 is excluded from the domain.

$(-\infty, -3) \cup (-3, \infty)$

Example: Let f and g be two functions defined as $f(x) = 2x$ and $g(x) = x^2 - 4$. Find the following and determine the domain of each:

$(f+g)(x) = f(x)+g(x) = 2x + x^2 - 4 = x^2 + 2x - 4$

domain = $(-\infty, \infty)$

$(f-g)(x) = f(x) - g(x) = 2x - (x^2-4) = 2x - x^2 + 4 = -x^2 + 2x + 4$

domain = $(-\infty, \infty)$

$(fg)(x) = f(x)\,g(x) = 2x(x^2-4) = 2x^3 - 8x$

domain = $(-\infty, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x}{x^2-4}$

domain : $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$x^2 - 4 \neq 0$

$(x+2)(x-2) \neq 0$

$x + 2 \neq 0 \qquad x - 2 \neq 0$

$x \neq -2 \qquad x \neq 2$

Example: Let $f(x) = -x^2 + x$ and $g(x) = -3x + 1$. Evaluate the following:

$(f+g)(1)$

$(f-g)(-3)$

$(fg)(2)$

$\left(\frac{f}{g}\right)(5)$

$f(x) = \frac{2}{x+1}$ $\quad$ $g(x) = -\frac{1}{x^2}$

$(f+g)(x) = f(x)+g(x) = \frac{2}{x+1} - \frac{1}{x^2}$

LCD $= x^2(x+1)$

$= \frac{2x^2}{x^2(x+1)} - \frac{x+1}{x^2(x+1)}$

Domain

$f(x) = \frac{2}{x+1}$ $\quad$ -1 is excluded

$g(x) = \frac{-1}{x^2}$ $\quad$ 0 is excluded

$= \frac{2x^2 - (x+1)}{x^2(x+1)}$

$= \frac{2x^2 - x - 1}{x^2(x+1)}$

Domain: -1 and 0 are excluded

$(-\infty,-1) \cup (-1,0) \cup (0,\infty)$

Composite Functions

Given two functions f and g, the composite function, denoted by $f°g$, is defined by

$$(f°g)(x) = \underbrace{f(g(x))}_{\text{"f of g of x"}}$$

To evaluate $(f°g)(4)$,

1. Rewrite $(f°g)(4)$ as $f(g(4))$.
1. Find $g(4)$.
2. Take the value of $g(4)$and plug it into $f(x)$

Example: If $f(x) = x + 1$, and $g(x) = x^2$, then

Work from the inside out.

Find $(f°g)(4)$.

$$1.\ (f \circ g)(4) = f(\underbrace{g(4)}_{g(4) = 4^2 = 16}) = f(16) = 16 + 1 = 17$$

Find $(f°g)(2)$. $= f(\underbrace{g(2)}_{g(2) = 2^2 = 4}) = f(4) = 4 + 1 = 5$

$$(g \circ f)(2) = g(\underbrace{f(2)}_{f(2) = 2+1 = 3}) = g(3) = 3^2 = 9$$

Find $(f \circ g)(x) = f(g(x)) = f(x^2) = \underline{x^2 + 1}$

Since $f(x) = x + 1$

Example: Suppose $f(x) = 2x^2$ and $g(x) = 1 - 3x^2$.

Find $(f \circ g)(3)$.

Find $(f \circ g)(2)$.

Find $(f \circ g)(-1)$.

Find $(f \circ g)(-2)$.

Example: Find expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$.

$f(x) = 2x + 5; g(x) = 3x^2$

$(f \circ g)(x) = f(g(x)) = f(3x^2) = 2(3x^2) + 5 = 6x^2 + 5$

$(g \circ f)(x) = g(f(x)) = g(2x+5) = 3(2x+5)^2$

$= 3(2x+5)(2x+5)$

$= (6x+15)(2x+5)$

$= 12x^2 + 30x + 30x + 75$

$= 12x^2 + 60x + 75$

$f(x) = |x|; g(x) = 3x - 4$

$(f \circ g)(x) = f(g(x)) = f(3x-4)$

$= |3x-4|$

$(g \circ f)(x) = g(f(x)) = g(|x|)$

$= 3|x| - 4$

$f(x) = \frac{1}{x^3}; g(x) = 2x + 5$

$f(x) = x - 2; g(x) = \sqrt{1-x}.$

The Difference Quotient: $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$

Example: Find the difference quotient of $f(x) = -3x + 1$

$f(x+h) = -3(x+h) + 1 = -3x - 3h + 1$

difference quotient: $\frac{-3x-3h+1-(-3x+1)}{h}$

$= \frac{\cancel{-3x}-3h+\cancel{1}+\cancel{3x}\cancel{-1}}{h}$

$= \frac{-3\cancel{h}}{\cancel{h}} = -3$

Example: Find the difference quotient of $f(x) = x^2 + 5x - 1$

Homework: Section 2.2: 7-13 (odd), 17-25 (odd), 49-59 (odd), 67-77 (odd), 105, 107

For problems 67-77, you don't need to find the domains.