Section 3.5
Equation That Are Reducible to Quadratic Form; Rational and Radical Equation

Objectives

- Solve polynomial equations by reducing them to quadratic form
- Solve equations containing rational expressions
- Solve equations containing radical expressions

Solving Equations by Reducing Them to Quadratic Form

Example: Solve the equation $3x^4 + 5x^2 - 2 = 0$

$\boxed{\text{Note: If } u = x^2 \quad \text{Then } u^2 = x^4}$

Substitute $u = x^2$

$3u^2 + 5u - 2 = 0$

Factor

$(3u - 1)(u + 2) = 0$

$3u - 1 = 0$ or $u + 2 = 0$

$+1 \quad +1$

$3u = 1$

$u = \frac{1}{3}$

Since $u = x^2$

$x^2 = \frac{1}{3}$

$x = \pm \frac{\sqrt{3}}{3}$

$u = -2$

Since $u = x^2$

$x^2 = -2$

$x = \pm i\sqrt{2}$

Example: Solve the equation $2x^4 + x^2 - 3 = 0$

Let $u = x^2$
Then $u^2 = x^4$

$2u^2 + u - 3 = 0$

$(2u + 3)(u - 1) = 0$

$2u + 3 = 0$ or $u - 1 = 0$

$-3 \quad -3$

$u = 1$

Since $u = x^2$

$x^2 = 1$

$x = \pm 1$

$2u = -3$ or

$u = -\frac{3}{2}$

Since $u = x^2$

$x^2 = -\frac{3}{2}$

$\sqrt{x^2} = \pm\sqrt{-\frac{3}{2}}$

$x = \pm i \frac{\sqrt{3}}{\sqrt{2}}$

rationalize the denominator

$x = \pm \frac{\sqrt{6}}{2} i$

Example: Solve the equation $x^6 - x^3 = 2$

Let $u = x^3$

$u^2 = x^6$

$u^2 - u = 2$

move everything to one side.

$u^2 - u - 2 = 0$

$(u-2)(u+1) = 0$

$u = 2$ or $u = -1$

$x^3 = 2$ or $x^3 = -1$

$x = \sqrt[3]{2}$ or $x = -1$

Example: Solve the equation $3x^6 + 5x^3 = 2$

$u = x^3$

$u^2 = x^6$

$3u^2 + 5u = 2$

$3u^2 + 5u - 2 = 0$

$(3u - 1)(u + 2) = 0$

$3u - 1 = 0$ or $u + 2 = 0$

$3u = 1$

$u = \frac{1}{3}$ or $u = -2$

$x^3 = \frac{1}{3}$

$x = \sqrt[3]{\frac{1}{3}}$ $\qquad$ $x = \sqrt[3]{-2}$

Solve Equations Containing Rational Expressions

Example: Solve $\frac{1}{4} - \frac{3}{2} = \frac{3}{x}$ LCD = $4x$

$$4x \cdot \frac{1}{4} - 4x \cdot \frac{3}{2} = 4x \cdot \frac{3}{x}$$

$$\cancel{4x} \cdot \frac{1}{\cancel{4}} - \overset{2}{\cancel{4}}x \cdot \frac{3}{\cancel{2}} = 4\cancel{x} \cdot \frac{3}{\cancel{x}}$$

$$x - 6x = 12$$

$$-5x = 12$$

$$x = -\frac{12}{5}$$

Example: Solve $-\frac{2}{3x} + \frac{1}{x} = \frac{1}{4}$ LCD = $12x$

$$12x\left(\frac{2}{3x}\right) + 12x\left(\frac{1}{x}\right) = 12x\left(\frac{1}{4}\right)$$

$$\overset{4}{\cancel{12x}}\left(\frac{2}{\cancel{3x}}\right) + 12\cancel{x}\left(\frac{1}{\cancel{x}}\right) = \overset{3}{\cancel{12}}x\left(\frac{1}{\cancel{4}}\right)$$

$$8 + 12 = 3x$$

$$20 = 3x$$

$$\frac{20}{3} = x$$

Example: $\frac{1}{x^2} - \frac{3}{x} = 10$ LCD = x^2

$$x^2\left(\frac{1}{x^2}\right) - x^2\left(\frac{3}{x}\right) = x^2 \cdot 10$$

$$\cancel{x^2}\left(\frac{1}{\cancel{x^2}}\right) - \overset{x}{\cancel{x^2}}\left(\frac{3}{\cancel{x}}\right) = x^2 \cdot 10$$

$$1 - 3x = 10x^2$$

Quadratic equation: move everything to one side.

$$0 = 10x^2 + 3x - 1$$

$$0 = (5x - 1)(2x + 1)$$

$$5x - 1 = 0 \quad \text{or} \quad 2x + 1 = 0$$

$$x = \frac{1}{5} \quad \text{or} \quad x = \frac{-1}{2}$$

Example: Solve $\frac{1}{x} = \frac{2}{x-2} + 3$ LCD = $x(x-2)$

$$x(x-2)\cdot\frac{1}{x} = x(x-2)\,\frac{2}{x-2} + x(x-2)\cdot 3$$

$$\cancel{x}(x-2)\cdot\frac{1}{\cancel{x}} = x\cancel{(x-2)}\,\frac{2}{\cancel{x-2}} + x(x-2)\cdot 3$$

$$x-2 = 2x + 3x(x-2)$$

$$x-2 = 2x + 3x^2 - 6x$$

$$0 = 3x^2 - 5x + 2$$

Quadratic equation: move everything to one side.

$$0 = (3x+1)(x-2)$$

$x = -1/3$ or $x = 2$

Example: Solve $\frac{2}{x+3} = \frac{1}{x} - \frac{3}{2}$ LCD = $2x(x+3)$

$$2x(x+3)\cdot\frac{2}{x+3} = 2x(x+3)\,\frac{1}{x} - 2x(x+3)\cdot\frac{3}{2}$$

$$2x\cancel{(x+3)}\cdot\frac{2}{\cancel{x+3}} = 2\cancel{x}(x+3)\,\frac{1}{\cancel{x}} - \cancel{2}x(x+3)\cdot\frac{3}{\cancel{2}}$$

$$4x = 2(x+3) - 3(x+3)$$

$$4x = 2x+6 - 3x-9$$

$$4x = -x-3$$

Linear equation: Solve for x

$$5x = -3$$

$$x = -\frac{3}{5}$$

Example: Solve $\frac{1}{\underbrace{x^2+4x-1}_{\text{Factor as } (x+5)(x-1)}} + \frac{6}{x+5} = \frac{1}{x-1}$ LCD = $(x+5)(x-1)$

$$(x+5)(x-1)\cdot\frac{1}{(x+5)(x-1)} + (x+5)(x-1)\,\frac{6}{x+5} = (x+5)(x-1)\cdot\frac{1}{x-1}$$

$$\cancel{(x+5)}\cancel{(x-1)}\cdot\frac{1}{\cancel{(x+5)}\cancel{(x-1)}} + \cancel{(x+5)}(x-1)\,\frac{6}{\cancel{x+5}} = (x+5)\cancel{(x-1)}\cdot\frac{1}{\cancel{x-1}}$$

$$1 + 6(x-1) = x+5$$

$$1 + 6x - 6 = x+5$$

Linear equation: solve for x.

$$6x - 5 = x + 5$$

$$\quad -x \qquad\quad -x$$

$$5x - 5 = 5$$

$$\quad +5 \qquad +5$$

$$5x = 10$$
$$x = 2$$

Solving Equations Containing Radical Expressions

Example: Solve $\sqrt{3x+1} + 2 = x - 1$

$$-2 \qquad -2$$

$$\sqrt{3x+1} = x - 3$$ square both sides

$$3x+1 = (x-3)^2$$

$$3x+1 = x^2 - 6x + 9$$ Quadratic equation: move everything to one side.

$$0 = x^2 - 9x + 8$$

$$0 = (x-8)(x-1)$$

$$x - 8 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = 8 \quad \text{or} \quad x = 1$$

Example: Solve $\sqrt{4x+5} - 1 = x + 1$

$$+1 \qquad +1$$

$$\sqrt{4x+5} = x + 2$$ square root

$$4x+5 = (x+2)^2$$

$$4x+5 = x^2 + 4x + 4$$

$$0 = x^2 - 1$$

$$0 = (x+1)(x-1)$$

$$x = \pm 1$$

Solving an Equation Containing Two Radicals

Example: Solve $\sqrt{3x+1} - \sqrt{x+4} = 1$

Isolate one radical
square both sides

$$\sqrt{3x+1} = 1 + \sqrt{x+4}$$ square both sides

$$3x+1 = \left(1 + \sqrt{x+4}\right)^2$$

$$3x+1 = 1 + 2\sqrt{x+4} + x + 4$$

$$3x+1 = 5 + x + 2\sqrt{x+4}$$ Isolate second radical.

$$-x-5 \qquad -5 \quad -x$$

$$\frac{2x-4}{2} = \frac{\cancel{2}\sqrt{x+4}}{\cancel{2}}$$

$x - 2 = \sqrt{x+4}$

$(x-2)^2 = (\sqrt{x+4})^2$ Square both sides

$x^2 - 4x + 4 = x + 4$ Quadratic equation: Move everything to one side.

$x^2 - 5x = 0$ Factor

$x(x-5) = 0$

$x = 0$ or $x - 5 = 0$

$x = 5$

Check: $x = 0$ does not work when you plug it back in. It is an extraneous solution.

Solution: $x = 5$

Homework: Section 3.5: 7-17 (odd), 19-49 (odd)**, skip 31**