Section 2.1
Functions

Objectives

1. Determine Whether a Relation Represents a Function
2. Find the Value of a Function
3. Find the Domain of a Function
4. Form the Sum, Difference, Product, and Quotient of Two Functions

Relation—a correspondence between two sets.

Example: Suppose I make $20 per hour.

Hours worked — Income

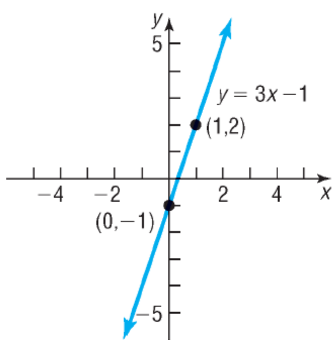
1 → 20
2 → 40
3 → 60

Example: Suppose I work for a company whose bonus plan is this: If the yearly profits are less than $10 million, there is no bonus. If the yearly profits are $10 million or more, all employees receive a $1000 bonus.

Profits — Bonus

5 → 0
8 → 0
13 → 1000
20 → 1000

The relation $y = 3x - 1$ shows a correspondence between x and y.

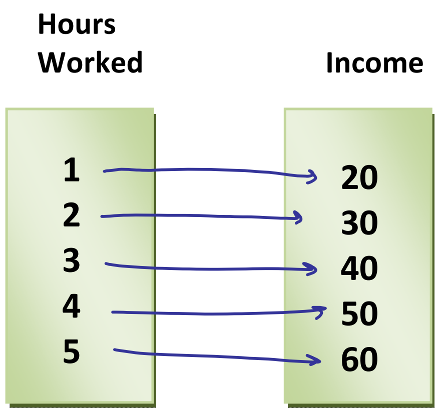


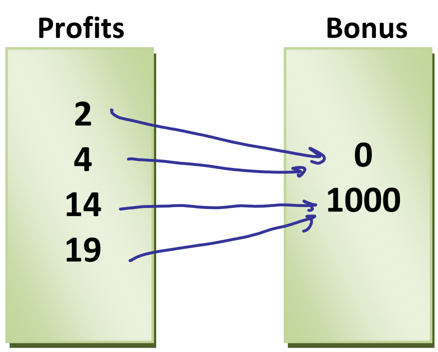
Function—a relation that associates with each possible value of x in with exactly one value of y.

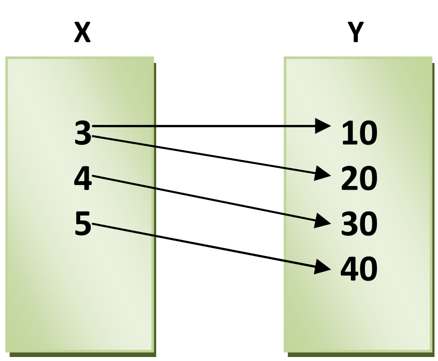
Domain—all possible values of x.

Range—all possible values of y.

You can use a mapping to determine whether a relation is a function.







Not a function because each element in the domain does not correspond to exactly one element in the range.

Determining Whether a Relation Represents a Function

Example: Determine whether the relation represents a function: {(-2, 5), (-1, 3), (3, 7), (4, 12)}

X	Y

$-2 \rightarrow 5$
$-1 \rightarrow 3$
$3 \rightarrow 7$
$4 \rightarrow 12$

Yes. Each value in x maps to only one value in y.

Example: Determine whether the relation represents a function: {(2, 4), (3, 4), (5, 6), (7, 10)}

X	Y

$2 \rightarrow 4$
$3 \rightarrow 4$
$5 \rightarrow 6$
$7 \rightarrow 10$

Yes. Each value in x maps to only one value in y. (Note it is ok for different values in X to map to the same Y value.)

Example: Determine whether the relation represents a function: {(1, 6), (1, 7), (2, -3), (3, 6)}

X	Y

$1 \rightarrow 6$
$1 \rightarrow 7$
$2 \rightarrow -3$
$3 \rightarrow 6$

No. One value in X maps to multiple values in Y.

Determining Whether an Equation is a Function

Example: Is $y = 3x - 1$ a function?

x	y
0	-1
1	2
2	5

yes. If you plug in a value for x, you'll get exactly 1 value for y.

Example: Is $y = \pm\sqrt{1 - 2x}$ a function?

x	y
0	1
0	-1

No. If you plug in a value for x, say x=0, you'll get two values for y.

Example: Is $x + y^2 = 1$ a function? No. If you plug in a value for x, say $x=0$, you'll get two values for y.

x	y
0	1
0	-1

$0 + y^2 = 1 \rightarrow y^2 = 1 \rightarrow y = \pm 1$

Example: Is $2x^2 + 2y^2 = 8$ a function?

Circle. Divide through by 2.

$x^2 + y^2 = 4$

x	y
0	2
0	-2

$0^2 + y^2 = 4 \rightarrow y^2 = 4 \rightarrow y = \pm 2$

no.

Find the Value of a Function

Functions are denoted by letters such as f, F, g, G and others.

Function notation: $f(x)$

This is read as "*f* of *x*" and is the value of the function *f* at the number *x*.

Example: For the function f defined by $f(x) = 20x$, evaluate

This function says "Take some input, x, and multiply it by 20."

a. $f(2) = 20 \cdot 2 = 40$

b. $f(10) = 20 \cdot 10 = 200$

Example: For the function f defined by $f(x) = 2x^2 + 3$, evaluate

← This function says "Take some input, x, square it, multiply the result by 2, and then add 3."

a. $f(5)$ $= 2(5)^2 + 3 = 50 + 3 = 53$

b. $f(-3)$ $= 2(-3)^2 + (-3) = 18 - 3 = 15$

c. $f(-x)$ $= 2(-x)^2 + 3 = 2x^2 + 3$

d. $f(x+3)$ $= 2(x+3)^2 + 3 = 2(x^2 + 6x + 9) + 3 = 2x^2 + 12x + 18 + 3$

$= 2x^2 + 12x + 21$

Implicit Form of a Function

When a function f is defined by an equation in x and y, we say that the function f is given <u>implicitly</u>.

Examples: $x^2 + y^2 = 25$

$2x + 3y = 19$

When it is possible to solve the equation for y in terms of x, then we write $y = f(x)$ and say that the function is given <u>explicitly</u>.

Examples:

$$\begin{array}{r} y + x^2 = 2 \\ -x^2 \quad -x^2 \\ \hline y = -x^2 + 2 \end{array}$$

← solved explicitly

$f(x) = -x^2 + 2$

Finding the Domain of a Function

Domain of f: The set of values that you're allowed to plug in for x.

When finding the domain of a function, remember two things:

1. You're not allowed to plug in values of x that would cause you to take the square root of a negative number.
2. You're not allowed to plug in values of x that would make the denominator of a fraction equal to zero.

Example: Find the domain of $y = \sqrt{x}$. ← you can't plug in a negative value for x.

Domain $= \{x \mid x \geq 0\}$ or $[0, \infty)$

Example: Find the domain of $\sqrt{2x - 1}$ ← The expression under the $\sqrt{\ }$ must be greater or equal to 0.

$$2x - 1 \geq 0$$
$$+1 \quad\quad +1$$
$$\frac{2x}{2} \geq \frac{1}{2}$$
$$x \geq \frac{1}{2}$$

domain $= \{x \mid x \geq \frac{1}{2}\}$ or $[\frac{1}{2}, \infty)$

Example: Find the domain of $\frac{x}{x-1}$ ← you can't plug in a value for x that makes the denominator equal to 0.

$$x - 1 \neq 0$$
$$+1 \quad\quad +1$$
$$x \neq 1$$

domain $= \{x \mid x \neq 1\}$

Example: Find the domain of $\frac{x^2}{x^2-4}$ ← you can't plug in a value for x that makes the denominator equal to 0.

$$x^2 - 4 \neq 0$$
$$(x+2)(x-2) \neq 0$$

$x + 2 \neq 0$ $x - 2 \neq 0$

$x \neq -2$ $x \neq 2$

domain $= \{x \mid x \neq -2, x \neq 2\}$

Example: Find the domain of $\frac{x+4}{x^3-4x}$

$x^3-4x \neq 0$

$x(x^2-4) \neq 0$

$x(x+2)(x-2) \neq 0$

$x \neq 0, \quad x \neq -2, \quad x \neq 2$ domain $= \{x \mid x \neq 0, x \neq -2, x \neq 2\}$

Example: Find the domain of $\frac{x}{\sqrt{x-4}}$

you can't plug in a value for x that makes the denominator equal 0 or the expression under the $\sqrt{\ }$ negative.

$\sqrt{x-4} \neq 0$ and $x-4 \geq 0$

$x \neq 4$ $\quad x \geq 4$

domain: $\{x \mid x > 4\}$ or $(4, \infty)$

Example: Find the domain of $\sqrt{\frac{2}{x+1}}$

$\frac{2}{x+1}$ will be > 0 if the denominator is positive

$x+1 \neq 0$

$x \neq -1$

$\frac{2}{x+1} \geq 0 \Rightarrow x+1 > 0$

$\Rightarrow x > -1$

domain $= \{x : x > -1\}$

or $(-1, \infty)$

Form the Sum, Difference, Product, and Quotient of Two Functions

Sum: $(f+g)(x) = f(x) + g(x)$

Difference: $(f-g)(x) = f(x) - g(x)$

Product: $(f \cdot g)(x) = f(x) \cdot g(x)$

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Example: Let f and g be two functions defined as $f(x) = 2x - 1$ and $g(x) = x + 3$. Find the following:

$f + g \;=\; 2x - 1 + x + 3 = 3x + 2$ Domain: All Reals

$f - g = 2x + 1 - (x + 3) = 2x + 1 - x - 3 = x - 2$ Domain: All Reals

$f \cdot g \;=\; (2x-1)(x+3) \overset{\text{FOIL}}{=} 2x^2 + 6x - x - 3 = 2x^2 + 5x - 3$

Domain: All Reals

$\frac{f}{g} \;=\; \frac{2x-1}{x+3}$ Domain: $\{x \mid x \neq -3\}$

Example: Let f and g be two functions defined as $f(x) = 2x$ and $g(x) = x^2 + 3$. Find the following:

$f + g \;=\; 2x + x^2 + 3 = x^2 + 2x + 3$

$f - g \;=\; 2x - (x^2 + 3) = 2x - x^2 - 3 = -x^2 + 2x - 3$

$f \cdot g \;=\; (2x)(x^2 + 3) \;=\; 2x^3 + 6x$

$\frac{f}{g} \;=\; \frac{2x}{x^2 + 3}$

The Difference Quotient: $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$

Example: Find the difference quotient of $f(x) = -3x + 1$

$$f(x+h) = -3(x+h)+1 = -3x-3h+1$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\overbrace{-3x-3h+1}^{f(x+h)} - \overbrace{(-3x+1)}^{f(x)}}{h}$$

$$= \frac{-3\cancel{x}-3h+\cancel{1}+3\cancel{x}-\cancel{1}}{h}$$

$$= -\frac{3h}{h} = -3$$

Example: Find the difference quotient of $f(x) = x^2 + 5x - 1$

$$f(x+h) = (x+h)^2 + 5(x+h) - 1$$

$$= \underbrace{(x+h)(x+h)}_{\text{FOIL}} + 5x + 5h - 1$$

$$= x^2 + hx + hx + h^2 + 5x + 5h - 1$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\overbrace{x^2+2hx+h^2+5x+5h-1}^{f(x+h)} - \overbrace{(x^2+5x-1)}^{f(x)}}{h}$$

$$= \frac{\cancel{x^2}+2hx+h^2+\cancel{5x}+5h-\cancel{1}-\cancel{x^2}-\cancel{5x}+\cancel{1}}{h}$$

$$= \frac{2hx+h^2+5h}{h}$$

$$= 2x+h+5$$