

Section 3.1

Quadratic Functions and Models

Objectives

- Graph a Quadratic Function Using Transformations
- Identify the Vertex and Axis of Symmetry of a Quadratic Function
- Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts
- Use the Maximum or Minimum Value of a Quadratic Function to Solve Applied Problems

A **quadratic function** is function of the form

$$f(x) = ax^2 + bx + c$$

highest power is 2

where a , b , and c are real numbers and $a \neq 0$.

Examples:

$$f(x) = x^2 - 2x - 5$$

$$f(x) = -2x^2 + 3x - 10$$

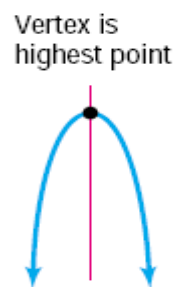
$$f(x) = x^2 + 3x$$

$$f(x) = -2x^2$$

Parabola—the graph of a quadratic function.



Vertex is lowest point
Opens up
 $a > 0$



Vertex is highest point
Axis of symmetry
Opens down
 $a < 0$

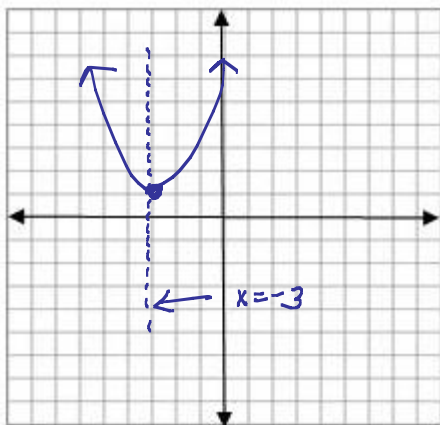
Another way to write a quadratic function is

$$f(x) = a(x - h)^2 + k$$

where (h, k) is the vertex of the parabola, and a either stretches (if $|a| > 1$) or compresses (if $0 < |a| < 1$) the parabola.

Also, if a is positive, the parabola opens up. If a is negative, the parabola opens down.

Example: Graph the function $f(x) = x^2 + 6x + 10$. Find the vertex and axis of symmetry.



Use the 'completing the square' method to put this function into $f(x) = a(x-h)^2 + k$ form.

$$f(x) = x^2 + 6x + 10 \quad (\text{leave space})$$

$$f(x) = x^2 + 6x + 9 + 10 - 9 \quad \leftarrow \text{Take } \frac{1}{2} \text{ of the coefficient of } x \text{ and square it, } 6 \div 2 = 3, 3^2 = 9$$

↓ Factor

$$f(x) = (x+3)(x+3) + 1$$

You must subtract it out again.

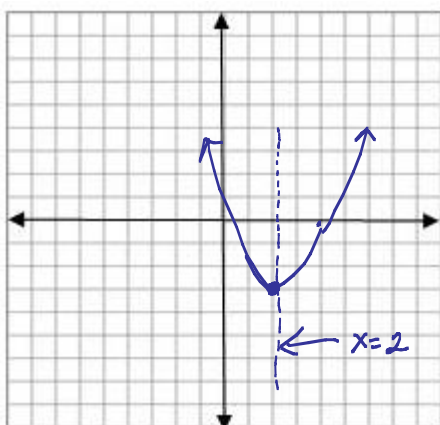
$$f(x) = (x+3)^2 + 1$$

$$h = -3, k = 1$$

$$\text{Vertex} = (-3, 1)$$

$$\text{Axis of symmetry} : x = -3$$

Example: Graph the function $f(x) = x^2 - 4x + 1$. Find the vertex and axis of symmetry.



$$f(x) = x^2 - 4x + 1$$

$$f(x) = x^2 - 4x + 4 + 1 - 4$$

$$f(x) = (x-2)(x-2) - 3$$

↓ Factor

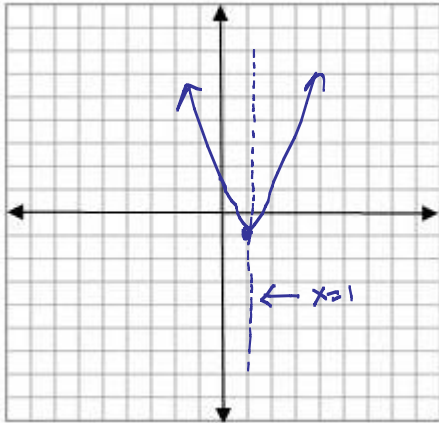
$$f(x) = (x-2)^2 - 3$$

$$\text{Vertex} = (2, -3)$$

$$\text{Axis of symmetry} : x = 2$$

Take $\frac{1}{2}$ of the coefficient of x and square it.

Example: Graph the function $f(x) = 2x^2 - 4x + 1$. Find the vertex and axis of symmetry.



To complete the square, the coefficient of x^2 must be 1.

Factor out 2 first.

$$f(x) = 2(x^2 - 2x) + 1$$

$$f(x) = 2(x^2 - 2x + 1) + 1 - 2$$

$$f(x) = 2(x-1)(x-1) - 1$$

$$f(x) = 2(x-1)^2 - 1$$

$$\text{Vertex} = (1, -1)$$

$$\text{Axis of symmetry: } x = 1$$

Note that we really added 2, since we factored out a 2.

Properties of the Graph of a Quadratic Function

Given $f(x) = ax^2 + bx + c$, then

1. The parabola opens up if a is a positive number, and opens down if a is a negative number.
2. The x -coordinate of the vertex is given by $-\frac{b}{2a}$
3. The y -coordinate of the vertex can be found by plugging in the x -coordinate (from step 2 above) into $f(x) = ax^2 + bx + c$.
4. Axis of symmetry: the line $x = -\frac{b}{2a}$

Example: Without graphing, locate the vertex and axis of symmetry of the parabola defined by $f(x) = 2x^2 - 4x + 1$.

$$\text{x-coordinate of vertex: } -\frac{b}{2a} = \frac{-(-4)}{2(2)} = \frac{4}{4} = 1$$

$$\text{y-coordinate of vertex: } f(1) = 2(1)^2 - 4(1) + 1 = 2 - 4 + 1 = -1$$

$$\text{Vertex: } (1, -1)$$

$$\text{Axis of symmetry: } (x = 1)$$

Example: Without graphing, locate the vertex and axis of symmetry of the parabola defined by $f(x) = x^2 + 10x - 6$.

$$\text{x-coordinate of vertex: } -\frac{b}{2a} = \frac{-10}{2(1)} = \frac{-10}{2} = -5$$

$$\text{y-coordinate of vertex: } f(-5) = (-5)^2 + 10(-5) - 6 = 25 - 50 - 6 = -31$$

$$\text{Vertex: } (-5, -31)$$

$$\text{Axis of symmetry: } x = -5$$

Intercepts of the Graph of a Quadratic Function

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

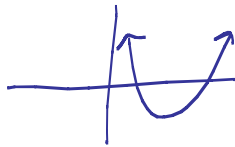
y-intercept: Set $x = 0$

$$f(0) = a(0)^2 + b(0) + c = c$$

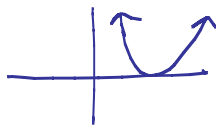
x-intercept(s): Set $f(x) = 0$ ($f(x) = y$)

$$0 = ax^2 + bx + c$$

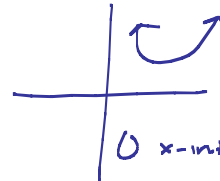
Try to factor or else
use the quadratic formula.



2 x-intercepts



1 x-intercept



0 x-intercepts

$$\text{Discriminant} = b^2 - 4ac$$

= expression under the
square root in the quadratic formula

1. When the discriminant is positive, there will be two x-intercepts.
2. When the discriminant is 0, there will be one x-intercept.
3. When the discriminant is negative, there will be no x-intercepts.

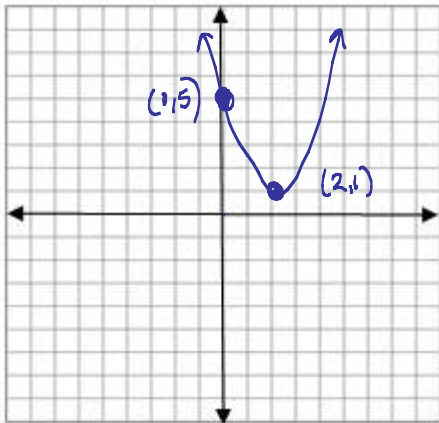
Finding a Quadratic Function Given the Vertex and a Point

Given the vertex (h, k) and one additional point on the graph of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, we can use

$$f(x) = a(x - h)^2 + k$$

to obtain the quadratic function.

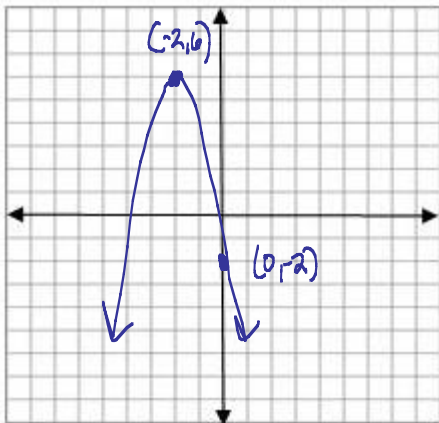
Example: Determine the quadratic function whose vertex is $(2, 1)$ and whose y-intercept is 5.



$$\begin{aligned} \text{Vertex} &= (2, 1) \\ \text{Intercept} &= (0, 5) \\ h &= 2 \\ k &= 1 \\ x &= 0 & \leftarrow \text{when } x=0, \text{ } f(x)=5 \\ f(x) &= 5 \end{aligned}$$

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ 5 &= a(0-2)^2 + 1 \\ 5 &= a(-2)^2 + 1 \\ \frac{-1}{4} &= 4a & \frac{-1}{1} \\ 1 &= a & f(x) = 1(x-2)^2 + 1 \end{aligned}$$

Example: Determine the quadratic function whose vertex is $(-2, 6)$ and whose y-intercept is -2.



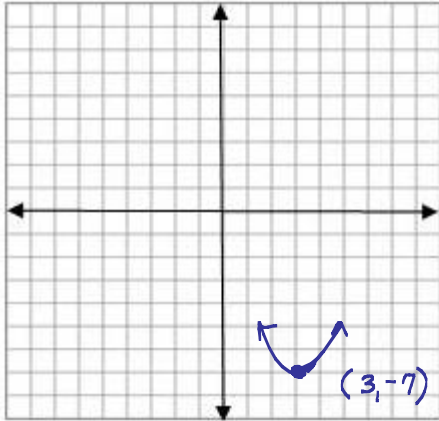
$$\begin{aligned} h &= -2 \\ k &= 6 \\ x &= 0 \\ f(x) &= -2 \\ -2 &= a(0+2)^2 + 6 \\ -2 &= a(2)^2 + 6 \\ -2 &= 4a + 6 \\ \frac{-8}{4} &= \frac{4a}{4} \\ -2 &= a \\ f(x) &= -2(x+2)^2 + 6 \end{aligned}$$

Determining If a Quadratic Function Has a Maximum or Minimum

Example: Determine if the quadratic function

$$f(x) = x^2 - 6x + 2$$

has a minimum or maximum value. Then find the minimum or maximum value.



$$\begin{aligned} a &= 1 \\ b &= -6 \\ c &= 2 \end{aligned}$$

Since $a > 0$, the parabola opens up.

$$\text{Vertex: } x\text{-coord.} = \frac{-b}{2a} = \frac{-(-6)}{2(1)} = \frac{6}{2} = 3$$

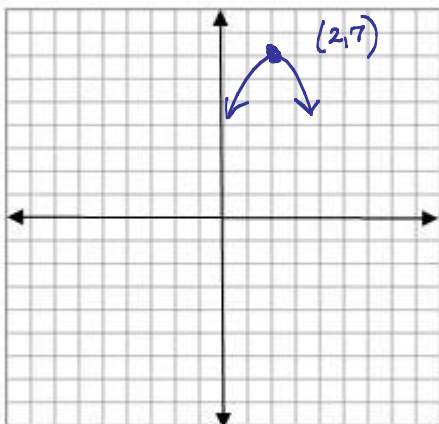
$$\begin{aligned} y\text{-coord.} &= f(3) = 3^2 - 6(3) + 2 \\ &= 9 - 18 + 2 = -7 \end{aligned}$$

The minimum value is -7 and occurs at $x = 3$.

Example: Determine if the quadratic function

$$f(x) = -2x^2 + 8x - 1$$

has a minimum or maximum value. Then find the minimum or maximum value.



$$\begin{aligned} a &= -2 \\ b &= 8 \\ c &= -1 \end{aligned}$$

Since $a < 0$, the parabola opens down.

$$\text{Vertex: } x\text{-coord.} = \frac{-b}{2a} = \frac{-8}{2(-2)} = \frac{-8}{-4} = 2$$

$$\begin{aligned} y\text{-coord.} &= f(2) = -2(2)^2 + 8(2) - 1 \\ &= 7 \end{aligned}$$

The maximum value is 7 and occurs at $x = 2$.