Section A-6
Complex Numbers; Quadratic Equations in the Complex Number System

Objectives
1. Add, Subtract, Multiply, and Divide Complex Numbers
2. Solve Quadratic Equations with a Negative Discriminant

Imaginary unit—a number denoted by i and whose square is -1.

$$i^2 = -1.$$

Complex numbers are numbers of the form $a + bi$, where a and b are real numbers.

Examples: $3 + 2i$ $5 - 3i$ $3i$

real part   imaginary part

since you could write this as $0 + 3i$

Adding and Subtracting Complex Numbers

Add or subtract the real parts, then add or subtract the imaginary parts.

Example: Write the expression in standard form: $(4 + 5i) + (2 - 3i)$.

$= 6 + 2i$

Example: Write the expression in standard form: $(-3 + 2i) + (4 - 7i)$.

$= 1 - 5i$

Example: Write the expression in standard form: $(4+5i)-(1+3i)$.

$= 4+5i-1-3i$

$= 3-2i$

Example: Write the expression in standard form: $(-10+2i)-(4-7i)$.

$= -10+2i-4+7i$

$= -14+9i$

Multiplying Complex Numbers

Example: Write the expression in standard form: $(5+3i)\cdot(4+2i)$.

FOIL

$= 20+10i+12i+6i^2$

$= 20+22i+6i^2$

$= 20+22i+6(-1)$ because $i^2=-1$

$= 20+22i-6$

$= 14+22i$

Example: Write the expression in standard form: $(6-2i)\cdot(10+3i)$.

$= 60+18i-20i-6i^2$

$= 60-2i-6(-1)$

$= 60-2i+6$

$= 66-2i$

Example: Write the expression in standard form: $(3+2i)\cdot(3-2i)$.

conjugate

When you multiply a complex number by its conjugate, the outer and inner products will cancel.

$= 9-6i+6i-4i^2$

$= 9-4i^2$

$= 9-4(-1)$

$= 9+4$

$= 13$

Conjugates

The conjugate of $a + bi$ is $a - bi$. The conjugate of $a - bi$ is $a + bi$.

number	conjugate
$4+3i$	$4-3i$
$-5+2i$	$-5-2i$
$10-4i$	$10+4i$

Example: Write the expression in standard form: $(4 + 3i) \cdot (4 - 3i)$.

Outer and inner products will cancel.

$$= 16 - \cancel{12i} + \cancel{12i} - 9i^2$$
$$= 16 - 9i^2$$
$$= 16 - 9(-1)$$
$$= 16 + 9$$
$$= 25$$

Example: Write the expression in standard form: $(5 + 2i) \cdot (5 - 2i)$.

$$= 25 - 4i^2$$
$$= 25 - 4(-1)$$
$$= 25 + 4$$
$$= 29$$

In general, $(a + bi) \cdot (a - bi) = a^2 + b^2$

For the previous problem,

$$(\underset{a}{5} + \underset{b}{2}i)(5 - 2i) = 5^2 + 2^2 = 29$$

Rationalizing denominators

It is not considered "good form" to have a complex number in the denominator of a fraction. Rewrite such fractions by rationalizing the denominator.

To rationalize the denominator, multiply the numerator and denominator by the conjugate.

Example: Write the expression in standard form: $\frac{2}{5+3i}$

$$\frac{2}{5+3i} = \frac{2}{5+3i} \cdot \frac{5-3i}{5-3i} = \frac{2(5-3i)}{25+9} = \frac{2(5-3i)}{34} = \frac{5-3i}{17} = \frac{5}{17} - \frac{3}{17}i$$

multiply by conjugate

a^2+b^2

cancel

standard form

Example: Write the expression in standard form: $\frac{10}{3-2i}$

$$\frac{10}{3-2i} = \frac{10}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{10(3+2i)}{9+4} = \frac{10(3+2i)}{13} = \frac{30+20i}{13} = \frac{30}{13} + \frac{20}{13}i$$

multiply by conjugate

a^2+b^2

Example: Write the expression in standard form: $\frac{10}{i}$.

$$\frac{10}{i} = \frac{10}{i} \cdot \frac{i}{i} = \frac{10i}{i^2} = \frac{10i}{-1} = -10i$$

multiply by conjugate

Powers of i

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$i^5 = i$
$i^6 = -1$
$i^7 = -1$
$i^8 = 1$

pattern repeats

$i^3 = i^2 \cdot i = (-1)(i) = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

The powers of i repeat with every fourth power. Therefore, when raising i to any integral power, the answer is always i, -1, $-i$ or 1.

To simplify powers of i, divide the exponent by 4.

If the remainder is 0, the answer is $i^0 = 1$.
If the remainder is 1, the answer is $i^1 = i$.
If the remainder is 2, the answer is $i^2 = -1$.
If the remainder is 3, the answer is $i^3 = -i$.

Example: $i^{14} = i^2 = -1$

$14 \div 4 = 3$ remainder 2

so $i^{14} = i^2$

Example: $i^{23} = i^3 = -i$

$23 \div 4 = 5$ remainder 3

so $i^{23} = i^3$

Example: $i^{27} = -i$

$27 \div 4 = 6$ remainder 3

so $i^{27} = i^3$

Principal Square Root of a Negative Number

If N is a positive real number, we define the the *principal square root* of N, denoted by $\sqrt{-N}$ as,

$$\sqrt{-N} = \sqrt{N}i$$

Examples: $\sqrt{-25} = 5i$

$\sqrt{-36} = 6i$

$\sqrt{37} = \sqrt{37}i$

Solve Quadratic Equations with a Negative Discriminant

Use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall: $b^2 - 4ac$ is called the discriminant.

If the discriminant is positive, the quadratic equation will have two real solutions.
If the discriminant is zero, the quadratic equation will have one real solution.
If the discriminant is negative, the quadratic equation will have two complex solutions.

Example: Solve the equation $x^2 - 2x + 5$ =0 in the complex number system.

$a = 1$

$b = -2$

$c = 5$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = \frac{2}{2} \pm \frac{4}{2} i = 1 \pm 2i$$

Example: Solve the equation $x^2 + 4x + 8 = 0$ in the complex number system.

$a = 1$

$b = 4$

$c = 8$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 32}}{2}$$

$$= \frac{-4 \pm \sqrt{-16}}{2}$$

$$= \frac{-4 \pm 4i}{2}$$

$$= -2 \pm 2i$$